AN APPROXIMATE THEORY FOR THE HEAD WAVE GENERATED BY A CYLINDRICAL INCLUSION IN A HOMOGENEOUS ELASTIC MEDIUM

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The study of longitudinal and transverse elastic waves in media which contain curvilinear boundaries is of great theoretical and practical significance.

However, up to the present time it has not been possible to obtain results which were sufficiently general to characterize the displacement field of elastic waves and which would shed light on their dynamic characteristics (amplitude, wave form, etc.).

The methods of solution of dynamic elasticity problems which have been worked out thus far have been applied to certain particular forms of curvilinear surfaces [1-8].

In the case of diffraction of steady-state elastic waves around an arbitrary rigidly embedded obstacle, the problem leads to a system of singular integral equations which can be reduced to a regular Fredholm system [1].

The two-dimensional problem of the diffraction of an elastic wave for a piecewise smooth contour was examined in [9]. This problem also leads to a system of integral equations. The solvability of the system was not demonstrated in this work. The three-dimensional diffraction problem was solved analogously in [10]; moreover, in contrast to [9], the solvability of the system of integral equations was proved. Because of significant mathematical difficulties, the results obtained in [1-10] do not allow one to deduce any physical consequences of elastic wave dynamics.

In a series of works contained in [11] an infinite series analysis was proposed for the representation of the solution of the elastic wave propagation problem in cylindrical and spherical regions. However, this method has likewise been applied only to particular forms of surfaces.

In [12] a sufficiently general method of solution of the problem of scattering of elastic waves by an arbitrary curvilinear surface was expounded. Corresponding asymptotic formulas were obtained for the reflection of longitudinal and transverse waves, and likewise for the head wave which is generated by the incidence of a plane transverse elastic wave on a curvilinear surface. This method is based on the application of a principle of Kirchhoff for a system of wave equations and a "principle of the isolation of an element" which has been applied to rectilinear [13, 14] and curvilinear surfaces [15].

Recently it has been shown that to the first order it is valid to use the "principle of the isolation of an element" to obtain the laws of reflection and refraction, both for rectilinear and curvilinear boundaries [13, 14], that is, for angles that are less than the limiting angle (angle of complete internal reflection). There is reason to suppose that the principle remains valid for angles larger than the limiting angle. This stems from the formulas obtained on the basis of an asymptotic (or ray) method [17, 18].

The head wave which is generated by the incidence of a plane elastic wave on a rigidly embedded circular cylinder is investigated below; the effects which arise in this case are investigated only in the exposed region (in accordance with Kirchhoff's principle). From a kinematic analysis it follows that the head wave will not be generated in the "shadow" region. In the "shadow" region there exists a more complicated diffraction picture that is not investigated. The analysis of the diffraction picture in the region of "shadow" or "half-shadow" is possible only on the basis of an exact solution and the application of methods which are analogous to those developed by Fok for the electrodynamic case [19, 20].

1. Formulation of the problem and construction of the solution. Let a plane transverse elastic wave be incident, in a direction parallel to the x-axis, on a cylinder of radius h placed in an infinite elastic space

$$\psi_0(x,\ t) = e^{-i\omega t} e^{ik_2 x} \tag{1.1}$$

It is assumed that the plane of incidence is perpendicular to the cylinder's generators. An outline of the cylinder in the $z = 0$ plane is shown in Fig. 1.

In accordance with the "principle of the isolation of an element",

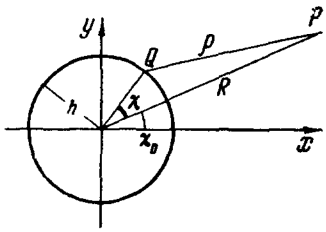

Fig. 1.

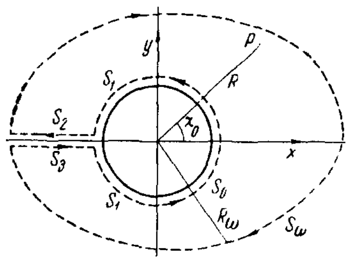

Fig. 2.

the reflection potentials near the boundary points of the cylinder have the forms

$$\varphi(x, y, t) = Ae^{-i\omega t}e^{ik_1(\alpha_1 x+\beta_1 y+\gamma_1)}, \qquad \psi_1(x, y, t) = Be^{-i\omega t}e^{ik_2(\alpha_2 x+\beta_2 y+\gamma_2)} \qquad (1.2)$$

The components of the displacement vector have, by (1.1) and (1.2), the form

$$u = \frac{\partial\varphi}{\partial x} + \frac{\partial\psi^*}{\partial y}, \qquad v = \frac{\partial\varphi}{\partial y} - \frac{\partial\psi^*}{\partial x} \qquad (\psi^* = \psi_0 + \psi_1) \qquad (1.3)$$

We denote by χ and χ_0 the angles (Fig. 1) formed by the radius vectors to the points Q and P and the x-axis.

On the boundary of the cylinder we have the following conditions:

$$u = v = 0 \qquad (1.4)$$

Using (1.1), (1.2), (1.3) and substituting into (1.4), we obtain

$$\begin{aligned}
&\alpha_1 = \varepsilon\sin^2\chi - \cos\chi\sqrt{1-\varepsilon^2\sin^2\chi}, && \alpha_2 = 1 - 2\cos^2\chi \\
&\beta_1 = -(\varepsilon\cos\chi \mp \sqrt{1-\varepsilon^2\sin^2\chi})\sin\chi, && \beta_2 = -2\sin\chi\cos\chi \qquad (1.5)\\
&\gamma_1 = h(\varepsilon\cos\chi + \sqrt{1-\varepsilon^2\sin^2\chi}), && \gamma_2 = 2h\cos\chi
\end{aligned}$$

$$\begin{aligned}
A(\chi) &= -\frac{2\varepsilon\sin\chi\cos\chi}{\varepsilon\sin^2\chi + \cos\chi\sqrt{1-\varepsilon^2\sin^2\chi}} \\
B(\chi) &= -\frac{\varepsilon\sin^2\chi - \cos\chi\sqrt{1-\varepsilon^2\sin^2\chi}}{\varepsilon\sin^2\chi + \cos\chi\sqrt{1-\varepsilon^2\sin^2\chi}}
\end{aligned} \qquad \left(\varepsilon = \frac{k_2}{k_1} > 1\right) \qquad (1.6)$$

It is required to find the value of the field at an arbitrary point P in the xy-space. We represent the reflection potential (1.2) at the arbitrary point P by means of the Kirchhoff formula

$$\varphi(P)=\frac{1}{4\pi}\int_S\left\{\varphi(Q)\frac{\partial}{\partial n}\frac{e^{ik_1r}}{r}-\frac{e^{ik_1r}}{r}\frac{\partial\varphi}{\partial n}\right\}dS$$
$$\psi_1(P)=\frac{1}{4\pi}\int_S\left\{\psi_1(Q)\frac{\partial}{\partial n}\frac{e^{ik_2r}}{r}-\frac{e^{ik_2r}}{r}\frac{\partial\psi_1}{\partial n}\right\}dS \tag{1.7}$$

In this, the surface of integration is to be understood as the sum $S = S_0 + S_1 + S_3 + S_\omega$ (Fig. 2). The integrals over S_1 are zero by virtue of the Kirchhoff condition ("shadow" region); the integrals on S_2 and S_3 mutually cancel. Calculations show [21] that the integrals on S_ω approach zero for $R_\omega \to \infty$. As a result of this the expressions entering into (1.7) have the form $dS_0 = hd_\chi d_\zeta$, $r = \sqrt{(\rho^2 + \zeta^2)}$. Assuming $h \ll R$, we may write

$$\rho=\sqrt{h^2+R^2-2hR\cos(\chi-\chi_0)}\approx R-h\cos(\chi-\chi_0)\approx R, \qquad -\frac{\pi}{2}\leqslant\chi\leqslant\frac{\pi}{2}$$

After some calculations analogous to [12], we obtain for the displacements in the reflected longitudinal wave

$$u_1(P)\approx ik_1{}^2\cos\chi_0U_1(\chi,R), \qquad v_1(P)\approx ik_1{}^2\sin\chi_0U_1(\chi,R)$$

Here

$$U_1(\chi,R)=\frac{h}{\sqrt{2\pi}}\frac{\exp\left[i(k_1R-{}^1/_4\pi)\right]}{\sqrt{k_1R}}\int_{-\pi/2}^{\pi/2}A(\chi)Q(\chi)\exp\left[ik_1f(\chi)\right]d\chi \tag{1.8}$$

For the displacements of the reflected transverse wave we obtain

$$u_2(P)\approx ik_2{}^2\sin\chi_0U_2(\chi,R), \qquad v_2(P)\approx -ik_2{}^2\cos\chi_0U_2(\chi,R)$$

Here

$$U_2(\chi,R)=\cos\frac{\chi_0}{2}\frac{h}{\sqrt{2\pi}}\frac{\exp\left[i(k_2R-{}^1/_4\pi)\right]}{\sqrt{k_2R}}\times$$
$$\times\int_{-\pi/2}^{\pi/2}B(\chi)\cos\left(\chi-\frac{\chi_0}{2}\right)\exp\left[-ip\cos\left(\chi-\frac{\chi_0}{2}\right)\right]d\chi \tag{1.9}$$

The quantities entering into (1.8) and (1.9) have the form

$$f(\chi)=(1-\varepsilon)\cos\chi-2\cos\frac{\chi_0}{2}\cos\left(\chi-\frac{\chi_0}{2}\right)$$
$$Q(\chi)=\cos\frac{\vartheta+\chi_k}{2}\cos\frac{\vartheta-\chi_k}{2}, \qquad p=2k_2h\cos\frac{\chi_0}{2}$$
$$\chi_k=\sin^{-1}(\varepsilon\sin\chi), \qquad \vartheta=\chi-\chi_0$$

The local coefficients of reflection $A(\chi)$ and $B(\chi)$ are determined by Formulas (1.6).

2. Approximate estimates of the integrals. Integrals of the type of (1.8) and (1.9) are most conveniently analysed by the method of stationary phase. To do this, we determine the singular points of the integrands in Expressions (1.8) and (1.9). Singularities can arise only from terms in the local coefficients of reflection (1.6), i.e. for $\sqrt{(1 - \epsilon^2 \sin^2 \chi)} = 0$.

In this case we have

$$\sin \chi^* = \theta \qquad \left(\theta = \frac{k_1}{k_2} < 1\right) \tag{2.1}$$

Thus, in the case of incidence of a transverse wave the branch points lie on the real axis. As regards the equation

$$\Delta = \varepsilon \sin^2 \chi + \cos \chi \sqrt{1 - \varepsilon^2 \sin^2 \chi} = 0 \tag{2.2}$$

an analysis shows that it has no real roots [12].

We turn next to the estimate of the integrals entering into (1.8) and (1.9). The equations of the saddle points have the form:

for the longitudinal reflected wave

$$\chi^{(1)} = \sin^{-1} \frac{\theta \sin \chi_0}{\sqrt{1 + 2\theta \cos \chi_0 + \theta^2}} \tag{2.3}$$

for the transverse reflected wave

$$\chi^{(2)} = \frac{1}{2} \chi_0 \tag{2.4}$$

It can be shown that in the sublimiting case there exists the relation $\chi^{(1)} < \chi^{(2)} < \chi^{(*)}$. We determine the saddle-point contours of integration for the calculation of the displacements of the reflected longitudinal and transverse waves.

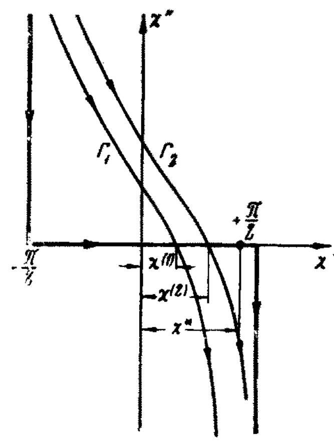

Fig. 3.

In the case of the longitudinal wave (1.8), the saddle-point path Γ_1 is

$$\operatorname{Im} f(\chi) = \operatorname{Im} f(\chi^{(1)}), \qquad \operatorname{Re} f(\chi) < 0$$

For a verification of the latter condition we make a change of variables

$$\chi = \chi^{(1)} + se^{i\theta_1} \qquad (s \ll 1)$$

As a result, if $f(\chi)$ is expanded in a Taylor series in the neighborhood of $\chi = \chi^{(1)}$, we obtain

$$\mathrm{Re}\, f(\chi) = -2s \sin\theta_1 (\chi^{(1)} + s\cos\theta_1) \frac{f''(\chi^{(1)})}{2} + \ldots$$

Hence, the behavior of the function Re $f(\chi)$ is determined by the sign of the second derivative $f''(\chi^{(1)})$. For considerations of simplicity let $\chi^{(1)} = 0$.

In this case

$$\mathrm{Re}\, f(\chi) = -2s^2 \sin\theta_1 \cos\theta_1 \frac{f''(0)}{2} = \frac{s^2}{2} \sin 2\theta^* f''(0) \qquad (\theta_1 = -\theta^*)$$

Assume that $f''(0) > 0$. Then the direction through the point $s = 0$, corresponding to $\theta^* = 1/4$, will be the most rapid [12] for the function $1/2\ s^2 \sin 2\theta^*$. Taking into account that

$$\theta_1 = -\theta^*, \qquad f''(\chi^{(1)}) = -\sqrt{1 + 2\theta\cos\chi_0 + \theta^2} < 0$$

we obtain the location of the saddle-point contour Γ_1 of the longitudinal wave (Fig. 3).

For the determination of the direction of the saddle-point path in the case of a transverse wave, we make the change of variables $\cos(\chi - 1/2\ \chi_0) = 1 + is^2$. Here s is a new real variable. Then the equation for the saddle-point contour in the complex plane (χ', χ'') will be

$$\cos(\chi' - {}^1/_2\, \chi_0) \cosh \chi'' = 1 \tag{2.5}$$

The saddle-point contour Γ_2 intersects the real axis at the point $\chi^{(2)} = 1/2\ \chi_0$ at an angle of 45° and approaches one side as $-1/2\ \pi + 1/2\ \chi_0 + i\infty$ and the other side as $+ 1/2\ \pi + 1/2\ \chi_0 - i\infty$ (Fig. 3).

We represent the displacements in the reflected longitudinal wave in the form

$$u_1(P) \approx ik_1 \cos\chi_0\, U_1(\chi^{(1)}, R), \qquad v_1(P) \approx ik_1 \sin\chi_0\, U_1(\chi^{(1)}, R)$$

Here

$$U_1(\chi^{(1)}, R) = \sqrt{\frac{h}{2R}}\, A(\chi^{(1)})\, Q(\chi^{(1)}) \exp[ik_1(R + hf(\chi^{(1)}))] \tag{2.6}$$

Analogously, for the displacements in the reflected transverse wave we have

$$u_2(P) \approx ik_2 \sin\chi_0 U_2(\chi_0, R), \qquad v_2(P) \approx -ik_2 \cos\chi_0 U_2(\chi_0, R)$$

Here

$$U_2(\chi_0, R) = \sqrt{\cos\frac{\chi_0}{2}\frac{h}{2R}}B\left(\frac{\chi_0}{2}\right)\exp\left[ik_2\left(R - 2h\cos\frac{\chi_0}{2}\right)\right] \qquad (2.7)$$

The formulas that have been obtained for the displacements of the reflected longitudinal and transverse waves may be easily derived by the

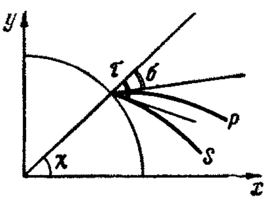

Fig. 4.

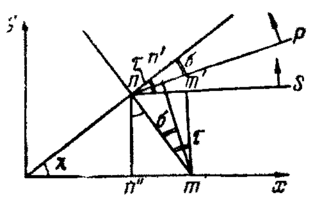

Fig. 5.

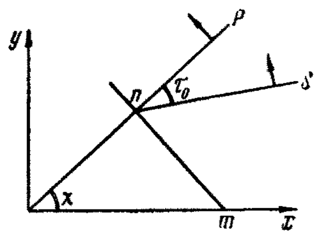

Fig. 6.

asymptotic (or ray) method. In the present case, this will be a consequence of an asymptotic analysis of the Kirchhoff integrals under the conditions $k_{1,2}R \gg 1$ and $k_{1,2}h \gg 1$. However, these formulas will be required for the subsequent investigation of the head wave.

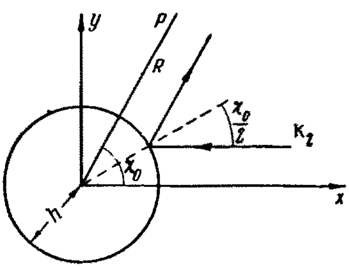

Fig. 7.

Expression (2.7) is not complete because it does not take into account the branch points in the integrands in (1.9). The latter must be considered in the deformation of the original contour of integration into the saddle-point contour Γ_2. We investigate this case in the following section.

3. Investigation of the head wave. We examine the generated head wave from a kinematic point of view. Figure 4 shows the ray pattern for the location of the fronts of the reflected longitudinal and transverse waves at some instant of time. We shall find the relationship linking the angles χ, τ and σ. From Fig. 5 we have $\cos\tau = \sin\chi$, $\cos\sigma = (a/b)\sin\chi$.

Under complete internal reflection it is clear that $\sigma = 0$ (Fig. 6), and we arrive at the condition corresponding to the emergence of a head wave

$$\sin \chi^* = b/a = \theta \tag{3.1}$$

As can be seen by comparison, conditions (2.1) and (3.1) are equivalent, i.e. the branch point geometrically determines the limiting angle θ^*.

If one now considers the path of the rays of the incident and reflected transverse waves, the pattern will be the following. The direction χ_0 = const corresponds to the point of observation P. The effects are examined at distances $R/\lambda \gg 1$; therefore, of all the rays scattered by the given point of the cylindrical surface, the dominant one will be in the direction χ_0 = const. The incident ray is parallel to the x-axis. In the approximation of the "principle of the isolation of an element" the well-known law of the equality of the angles of incidence and reflection must be maintained near the surface. Therefore the reflected ray of the transverse wave will match the angle $\chi^{(2)} = 1/2\ \chi_0$, i.e. the saddle-point $\chi^{(2)}$ determines the angle at which the transverse wave reflects in the neighborhood of the cylindrical surface (Fig. 7). At complete internal reflection we obtain

$$\chi^{(2)} = \chi^* = \sin^{-1}\theta \tag{3.2}$$

We examine a particular case of an elastic medium. We set $\lambda = \mu$ (Poisson hypothesis), then

$$\sin \chi^* = b/a = 1/\sqrt{3}, \quad \text{or} \quad \chi^* \approx 35°$$

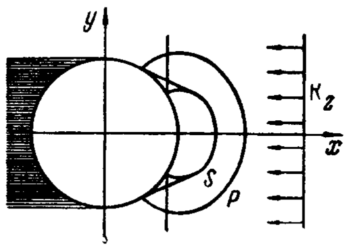

Fig. 8.

If it is assumed, moreover, that $b/a = 1/2$, then $\chi^* \approx 30°$, i.e. in both cases the head wave appears in the exposed part of the cylinder surface. However, for elastic media we always have $a > b$, hence the head wave cannot appear in the shadow region in the present formulation of the problem.*

Let $\chi^{(2)} = \chi^*$. This means that at a certain instant of time the front of the reflected longitudinal wave begins to overtake the reflected transverse wave front, breaking away from the latter and producing an

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* For $a = b$ (a fluid medium in which the head wave is absent in the given formulation of the problem) the angle $\chi^* \approx 90°$.

additional excitation – the head wave. The position of these fronts is shown in Fig. 8. As a result of this kinematic analysis we see that the complete transverse wave field is composed of two parts

$$U_i^* = u_i + u_i^* \qquad (i = 1, 2)$$

Here u_i is determined by Formula (2.7), and u_i^* is the integral over the cut Γ_0 (Fig. 9). We turn now to its analysis.

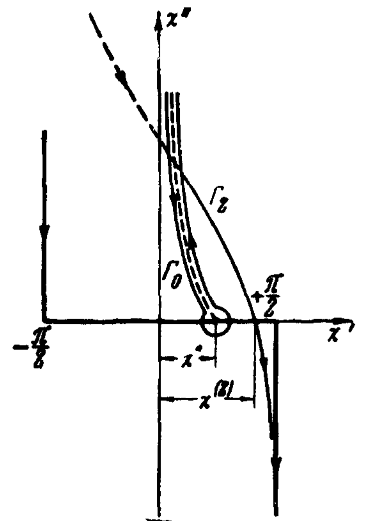

Fig. 9.

Because of the presence of the radical $\sqrt{(1 - \epsilon^2 \sin^2 \chi)}$ the integrand in (1.9) will be a double- valued function with a branch point $\chi^* = \sin^{-1}\theta$ lying on the real axis. As a consequence of this, the investigation of the integral entering into (1.9) is most conveniently carried out on a two-sheeted Riemann surface [22].

We introduce the final expressions for the displacements in the head wave

$$u^*(P) \approx ik_2^2 \sin\chi_0 U^*(\chi, R), \qquad v^*(P) \approx -ik_2^2 \cos\chi_0 U^*(\chi, R) \qquad (3.3)$$

$$U^*(\chi, R) = -4h \frac{\cos{}^1/_2\,\chi_0}{\sqrt{2\pi}} \frac{\exp[i(k_2R - {}^1/_4\,\pi)]}{\sqrt{k_2R}} \int_{\chi^*}^{i\infty} \Phi(\chi) \exp\left[-ip\cos\left(\chi - \frac{\chi_0}{2}\right)\right] d\chi$$

$$\Phi(\chi) = q\cos\left(\chi - \frac{\chi_0}{2}\right) \frac{\sin^2\chi}{\sin^4\chi - q^2} \qquad (q = \cos\chi\sqrt{\theta^2 - \sin^2\chi}) \qquad (3.4)$$

We make the change of variables $\beta = \chi^* - \chi$. Then, taking into account the smallness of β, (3.3) takes on the form

$$U^*(\beta, R) \approx -16h \frac{\cos(\chi^* - {}^1/_2\,\chi_0)}{\sqrt{\pi}\,(i\tan\chi^*)^{3/2}} \frac{\exp[i(k_2R - {}^1/_4\,\pi)]}{\sqrt{k_2R}} \int_0^{\chi^* - i\infty} e^{-ipf(\beta)} \left(i\tan\frac{\beta}{2}\right)^{3/2} \frac{d\beta}{2\sin{}^1/_2\,\beta} \qquad (3.5)$$

The integral that has been obtained, (3.5), can be expressed in terms of Weber functions. We use the Fok-Brekhovskikh equality [22]

$$\frac{1}{\Gamma(-n)} \int_0^{\pi/2 + \beta_0 - i\infty} e^{i/2\,(\xi^2 - \eta^2)\cos\beta - i\xi\eta\sin\beta} \left(i\tan\frac{\beta}{2}\right)^{-n} \frac{d\beta}{2\sin{}^1/_2\,\beta}$$

$$= D_n(\xi - i\xi)\, D_n(\eta + i\eta) \qquad (3.6)$$

We transform this equality. It is well known [23] that the Weber equation is not changed if n, ξ, η in this equation are simultaneously replaced by $-(n+1)$, $\pm i\xi$, $\pm i\eta$. Following this, (3.6) takes on the form

$$\frac{1}{\Gamma(n+1)}\int\limits_{0}^{\pi/2+\beta_0-i\infty}\exp-\left[\frac{i}{2}(\xi^2-\eta^2)\cos\beta-i\xi\eta\sin\beta\right]\left(i\tan\frac{\beta}{2}\right)^{n+1}\frac{d\beta}{2\sin^{1/2}\beta}$$
$$=D_{-(n+1)}(\xi+i\xi)\,D_{-(n+1)}(-\eta+i\eta) \tag{3.7}$$

Expression (3.5) may be transformed into (3.7) if we set

$$\xi=\sqrt{2p}\cos\frac{{}^1\!/_2\,\chi_0-\chi^*}{2},\qquad \eta=\sqrt{2p}\sin\frac{{}^1\!/_2\,\chi_0-\chi^*}{2}$$

Then we obtain

$$U^*(\xi,\eta,R)\approx-8h\frac{\cos({}^1\!/_2\,\chi_0-\chi^*)}{(i\tan\chi^*)^{3/2}}\frac{\exp[i(k_2R-{}^1\!/_4\,\pi)]}{\sqrt{k_2R}}D_{-3/2}(\xi+i\xi)\,D_{-3/2}(-\eta+i\eta)$$

Since $\xi \gg 1$, we may use the asymptotic expansion of the Weber function [23]

$$D_n(z)\approx z^n\exp\left(-\frac{1}{4}z^2\right)\left\{1-\frac{n(n-1)}{2z^2}+\frac{n(n-1)(n-2)(n-3)}{2\cdot 4\cdot z^2}+\dots\right\}$$

As a result we obtain

$$D_{-3/2}(\xi+i\xi)\approx\exp(-{}^1\!/_2\,i\xi^2)\,\frac{\exp(-{}^3\!/_8\,i\pi)}{\xi^{3/2}(\sqrt{2})^{3/2}} \tag{3.8}$$

Finally, the expressions for the displacements of the head wave take on the form

$$u^*(P)\approx\sin\chi_0U^*(\psi,R),\qquad v^*(P)\approx-\cos\chi_0U^*(\psi,R)$$

Here

$$U^*(\psi,R)=-\frac{(\sqrt{1-\theta^2})^{3/2}\cos\psi}{2^{3/4}\sqrt{Rh}}\,\frac{\exp\{ik_2[R-2h\cos{}^1\!/_2\,\chi_0\cos\psi+{}^1\!/_2\,\pi]\}}{L^{3/2}}\,F(\eta)$$

$$F(\eta)=\eta^{3/2}\exp\frac{-3i\pi}{8}\exp\frac{-i\eta^2}{2}D_{-3/2}(-\eta+i\eta),\qquad L=\theta\cos{}^1\!/_2\,\chi_0\sin\psi \tag{3.9}$$
$$\psi=\tfrac{1}{2}\chi_0-\chi^*$$

The function $F(\eta)$ can be expanded in an asymptotic series both for small and large η [22].

The formulas that have been obtained describe the displacement field in the head wave for $\chi_0/2 > \chi^*$, where $\chi^* = \sin^{-1}\theta$. We remark that in the case of a fluid medium ($\theta = 1$), $u^* = v^* = 0$, i.e. the head wave is absent.

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